Recall that given $f \in C^\alpha(\overline{B_1})$

$$u(x) = -\int_{B_1(0)} G(x,y) f(y) dy$$

is of $D^2(B_1)$ and $C^0(\overline{B_1})$.

Moreover, $\Delta u = f$ in B_1
$u \equiv 0$ on ∂B_1

(See definition of C^α in Note)

Q1. $u \in C^{2,\alpha}(\overline{B_1})$??

Q2. $\sup_{B_1} \|\nabla^2 u\|$??

$\|u\|_{C^2} \leq C$??

Thm) [Kellogg 1931]

Let $\Omega \subset \mathbb{R}^n$ open bounded w/ C^∞ $\partial\Omega$.

Given $\alpha \in (0,1)$, $f \in C^\alpha(\bar\Omega)$.

$\exists$ a unique $u \in C^{2,\alpha}(\bar\Omega)$ s.t.

$$\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

Moreover. $\|u\|_{C^{2,\alpha}(\bar\Omega)} \le C \|f\|_{C^\alpha(\bar\Omega)}$.

where $C = C(n, \Omega, \alpha)$

We won't prove this kellogg's thm in this course

For $B_1 = \Omega$, $u_{ij} = -\int_0 G_{ij}(f(y) - f(x))dy$
$- f(x) \int_{\partial B} G_i \nu_i d\sigma$.

See [GT] section 4. calculate $\|u\|_{C^{2,\alpha}}$ by ↓

Thm) Schauder estimate

$a_{ij}, b_i, c, f \in C^{\alpha}(\bar{\Omega})$, $g \in C^{2,\alpha}(\bar{\Omega})$

$\alpha \in (0,1)$, $a_{ij} = a_{ji}$,

$\Sigma a_{ij} \xi_i \xi_j \geq \lambda |\xi|^2$ for $\lambda > 0$.

Suppose that $\exists\, u \in C^{2,\alpha}(\bar{\Omega})$ s.t.

$$Lu = \Sigma a_{ij} u_{ij} + \Sigma b_i u_i + cu = f \quad \text{in } \Omega$$

$$u = g \quad \text{in } \partial\Omega.$$

Then. $\|u\|_{C^{2,\alpha}} \leq C(\|f\|_{C^{\alpha}} + \|g\|_{C^{2,\alpha}} + \sup_{\Omega}|u|)$

where $C = C(n, \alpha, \Omega, a, b, c)$

Thm) Existence of Energy minimizer.

Let $f, g \in C^\infty(\bar{\Omega})$

Then, $\exists\, u \in C^\infty(\bar{\Omega})$. s.t. $u = g$ on $\partial\Omega$

$$\int_\Omega \frac{1}{2}\|\nabla u\|^2 + fu \le \int_\Omega \frac{1}{2}\|\nabla v\|^2 + fv$$

holds for all $v \in C^\infty(\bar{\Omega})$. $v = g$ on $\partial\Omega$

In particular, by the calculus of variation. $\Delta u = f$ in Ω

$u = g$ on $\partial\Omega$

An indirect way to prove
the Kellogg's theorem.

Step 1) Given $f \in C^\alpha(\bar{\Omega})$

we find a seq $f_i \in C^\infty$ such that

$$\lim_{i \to +\infty} \|f - f_i\|_{C^\alpha} = 0$$

Step 2) By the existence of mini.

we have $u_i \in C^\infty(\bar{\Omega})$ s.t.

$$\Delta u_i = f_i \text{ in } \Omega. \quad u_i = 0 \text{ on } \partial\Omega$$

Step 3) Schauder estimate

$$\Rightarrow \|u_i\|_{C^{2,\alpha}} \le C\|f_i\|_{C^\alpha} \le C\|f\|_{C^\alpha} + \varepsilon$$

for $i \gg 1$.

Step 4) Arzela-Ascoli

$\Rightarrow \exists$ a subsequential limit u of u_i s.t.

$u \in C^{2,\alpha}$, $\lim \|u_{i_m} - u\|_{C^{2,\alpha}} = 0$ as $i_m \to +\infty$.

Ex 1) $f, b_i, c \in C^{\alpha}(\bar{\Omega})$, $g \in C^{2,\alpha}(\bar{\Omega})$,

$\alpha \in (0,1)$, Then, $\exists$ small $\varepsilon > 0$.

such that

if $\|f\|_{C^\alpha} + \|c\|_{C^\alpha} + \sum \|b_i\|_{C^\alpha} + \|g\|_{C^{2,\alpha}} < \varepsilon$.

then, $\exists u \in C^{2,\alpha}(\bar{\Omega})$. s.t.

$$\begin{aligned} \Delta u + \sum b_i u_i + cu &= f. \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

Pf) By the Kellogg's thm

$\exists v_0 \in C^{2,\alpha}$ s.t.

$$\Delta v_0 = f - \Delta g \in C^{\alpha}, \quad v_0 = 0 \text{ on } \partial\Omega.$$

Define $u_0 = g + v_0 \in C^{2,\alpha}$

Then. $\Delta u_0 = \Delta g + \Delta v_0 = f$ in Ω

$u_0 = g$ on $\partial\Omega$

By K. thm. $\exists v_1 \in C^{2,\alpha}$ s.t. $v_1 = 0$ on $\partial\Omega$.

$$\Delta v_1 = f_1 = -\sum b_i \partial_i u_0 - c u_0 \in C^\alpha$$

Define $u_1 = v_1 + u_0 \in C^{2,\alpha}$

$$\Rightarrow \Delta u_1 = \Delta v_1 + \Delta u_0 = -\sum b_i \partial_i u_0 - c u_0 + f$$

$$\therefore \Delta u_1 + \sum b_i \partial_i u_0 + c u_0 = f \quad \text{in } \Omega$$

$$u_1 = g \quad \text{on } \partial\Omega$$

Similarly. $\exists v_2 \in C^{2,\alpha}$ s.t. $v_2 = 0$ on $\partial\Omega$

$$\Delta v_2 = f_2 = -\sum b_i \partial_i v_1 - c v_1$$

Define $u_2 = v_2 + u_1$

$$\Rightarrow \Delta u_2 = f - \sum b_i \partial_i u_1 - c u_1$$

$$\therefore \Delta u_2 + \sum b_i \partial_i u_1 + c u_1 = f \quad \text{in } \Omega$$

$$u_2 = g \quad \text{on } \partial\Omega$$

We iterate this process so that we obtain $u_m \in C^{2,\alpha}$, $v_m \in C^{2,\alpha}$

$f_m \in C^\alpha$. $u_m = v_m + u_{m-1}$

$$f_{m+1} = -\sum b_i \partial_i v_m - c v_m.$$

$$\Delta v_m = f_m \text{ in } \Omega. \quad v_m = 0 \text{ on } \partial\Omega$$

$$\Delta u_m + \sum b_i \partial_i u_{m-1} + c u_{m-1} = f \text{ in } \Omega$$

$$u_m = g \text{ on } \partial\Omega$$

Claim: $\lim\limits_{m\to+\infty} \|u_{m+k} - u_m\|_{C^{2,\alpha}} = 0$

if ε is small enough.

If this claim is true, then

$u = \lim\limits_{m\to+\infty} u_m \in C^{2,\alpha}(\bar{\Omega})$, $\lim \|u - u_m\|_{C^{2,\alpha}} = 0$.

$f = \lim \Delta u_m + \sum b_i \partial_i u_{m-1} + c u_{m-1} = \Delta u + \sum b_i \partial_i u + cu$.

$u = g$ on $\partial\Omega$.

Let's prove the claim.

$$f_m = -\sum b_i \partial_i u_{m-1} - c u_{m-1}$$

$$\|f_m\|_{C^\alpha} \le \sum \|b_i \partial_i v_{m-1}\|_{C^\alpha} + \|c v_{m-1}\|_{C^\alpha}$$

$$\le \sum \|b_i\|_{C^\alpha} \|v_{m-1}\|_{C^{1,\alpha}} + \|c\|_{C^\alpha} \|v_{m-1}\|_{C^\alpha}$$

$(\because [fg]_\alpha \le [f]_\alpha \sup|g| + [g]_\alpha \sup|f|)$

$$\le \varepsilon \|v_{m-1}\|_{C^{2,\alpha}}.$$

By K's thm, & $\Delta v_m = f_m$ in Ω
$v_m = 0$ on $\partial\Omega$

we have $\|v_m\|_{C^{2,\alpha}} \le C_0 \|f_m\|_{C^\alpha}$
where $C_0 = C_0(n, \Omega)$

$$\therefore \|v_m\|_{C^{2,\alpha}} \le C_0 \varepsilon \|v_{m-1}\|_{C^{2,\alpha}}$$

By choosing ε small enough to have $C_0 \varepsilon \le 1/2$

Ex) Let $f \in C^{\alpha}$, $g \in C^{2,\alpha}$, $\alpha \in (0,1)$

$\exists\, \varepsilon > 0$ s.t. if $\|f\|_{C^{\alpha}} + \|g\|_{C^{2,\alpha}} \leq \varepsilon$.

then we have a $u \in C^{2,\alpha}(\bar{\Omega})$

satisfying
$$\frac{\Delta u}{1+u^2} = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

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pf)
$$f = \frac{\Delta u}{1+u^2} = \underbrace{\overset{\varepsilon}{\overset{\sim}{\Delta u}}}_{\text{Linearized operator}} - \underbrace{\frac{u^2 \Delta u}{1+u^2}}_{\text{Error}} \sim \varepsilon^3$$

$\exists\, v_0 \in C^{2,\alpha}$ s.t. $\Delta v_0 = f - \Delta g \in C^{\alpha}$

$v_0 = 0$ on $\partial\Omega$

Define

$u_0 = g + v_0 \in C^{2,\alpha}$

$$f_1 = \frac{u_0^2 \Delta u_0}{1+u_0^2} \in C^{\alpha}$$

$\exists\, v_1 \in C^{2,\alpha}$ s.t. $\Delta v_1 = f_1$ in Ω

$v_1 = 0$ on $\partial\Omega$

Define $u_1 = u_0 + v_1$

$\Rightarrow \Delta u_1 = \Delta u_0 + \Delta v_1 = \Delta g + \Delta u_0 + f_1$

$$= f + \frac{u_0^2 \Delta u_0}{1+u_0^2}$$

$$\Rightarrow \Delta u_1 - \frac{u_0^2 \Delta u_0}{1+u_0^2} = f \quad \text{in } \Omega$$

$$u_1 = g \quad \text{on } \partial\Omega$$

$$f_2 = \frac{u_1^2 \Delta u_1}{1+u_1^2} - \frac{u_0^2 \Delta u_0}{1+u_0^2} \in C^\alpha.$$

$\Delta v_2 = f_2$. in Ω. $v_2 = 0$ on $\partial\Omega$

$u_2 = v_2 + u_1$

$$\Rightarrow f = \Delta u_2 - \frac{u_1^2 \Delta u_1}{1+u_1^2}$$

$$f_{m+1} = \frac{u_m^2 \Delta u_m}{1+u_m^2} - \frac{u_{m-1}^2 \Delta u_{m-1}}{1+u_{m-1}^2}$$

$\Delta v_{m+1} = f_{m+1}$ in Ω. $v_{m+1} = 0$ on $\partial\Omega$

$u_{m+1} = v_{m+1} + u_m$

If $\lim_{m\to+\infty} \|u_{m+k} - u_m\|_{C^{2,\alpha}} = 0$

then. $u = \lim u_m \in C^{2,\alpha}$ satisfies

$$\Delta u - \frac{u^2 \Delta u}{1+u^2} = f \quad \text{in } \Omega$$

$$u = g \quad \text{on } \partial\Omega$$

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① By K' thm, $\|v_m\|_{C^{2,\alpha}} \leq C_0 \|f_m\|_{C^\alpha}$.

$$f_{m+1} = \frac{u_m^2 \Delta u_m}{1+u_m^2} - \frac{u_{m-1}^2 \Delta u_{m-1}}{1+u_{m-1}^2}$$

$$= \frac{u_m^2}{1+u_m^2}(\Delta u_m - \Delta u_{m-1}) + \Delta u_{m-1}\left(\frac{u_m^2}{1+u_m^2} - \frac{u_{m-1}^2}{1+u_{m-1}^2}\right)$$

$$= \frac{u_m^2}{1+u_m^2}\Delta V_m + \Delta u_{m-1}\frac{u_m^2 - u_{m-1}^2}{(1+u_m^2)(1+u_{m-1}^2)}$$

$$= \frac{u_m^2}{1+u_m^2}\Delta V_m + \frac{(\Delta u_{m-1})(u_m+u_{m-1})V_m}{(1+u_m^2)(1+u_{m-1}^2)}$$

$$\Rightarrow \|f_{m+1}\|_{C^\alpha} \le C\|V_m\|_{C^{2,\alpha}}\left(\|u_m\|_{C^{2,\alpha}} + \|u_{m-1}\|_{C^{2,\alpha}}\right)$$

$\Rightarrow \|f_{m+1}\|_{C^\alpha}$

$\leq C_1 \|V_m\|_{C^{2,\alpha}} \left(\|g\|_{C^{2,\alpha}} + \sum_{i=0}^{m} \|V_m\|_{C^{2,\alpha}} \right)$

$\exists\, C_2$ s.t.

$\|V_0\|_{C^{2,\alpha}} + \|g\|_{C^{2,\alpha}} \leq C_2 \varepsilon.$

$\|f_1\|_{C^\alpha} \leq C_2 \varepsilon^3.$

Let $M = 100 + C_0 + C_1 + C_2$

suppose $\varepsilon \leq M^{-4}.$

Inductively, we can obtain

$\|V_k\|_{C^{2,\alpha}} \leq M^{-3-k}.$

$\|g\|_{C^{2,\alpha}} + \sum_{i=0}^{k} \|V_i\|_{C^{2,\alpha}} \leq M^{-2}.$

$\Rightarrow \|U_{m+k} - U_m\|_{C^{2,\alpha}} \leq M^{-3-m} \to 0$

as $m \to +\infty$